Week 7 - Monday

COMP 2230

Last time

- Halting problem
- Cardinality
- Countable and uncountable infinity
- Started relations

Questions?

Assignment 3

Logical warmup

- All but one of the numbers from 1 to 100 are read to you, one every 10 seconds, but in no particular order.
- You have a good mind, but only a normal memory, and no means of recording information during the process.
- How can you ensure you can figure out which number was not called out?

Relations

Relations

- **Relations** are generalizations of functions
- In a function, an element of the domain must map to exactly one element of the co-domain
- In a relation, an element from one set can be related to any number (from zero up to infinity) of other elements
- Like functions, we're usually going to focus on binary relations
- We can define any binary relation between sets A and B as a subset of $A \times B$

Function or not?

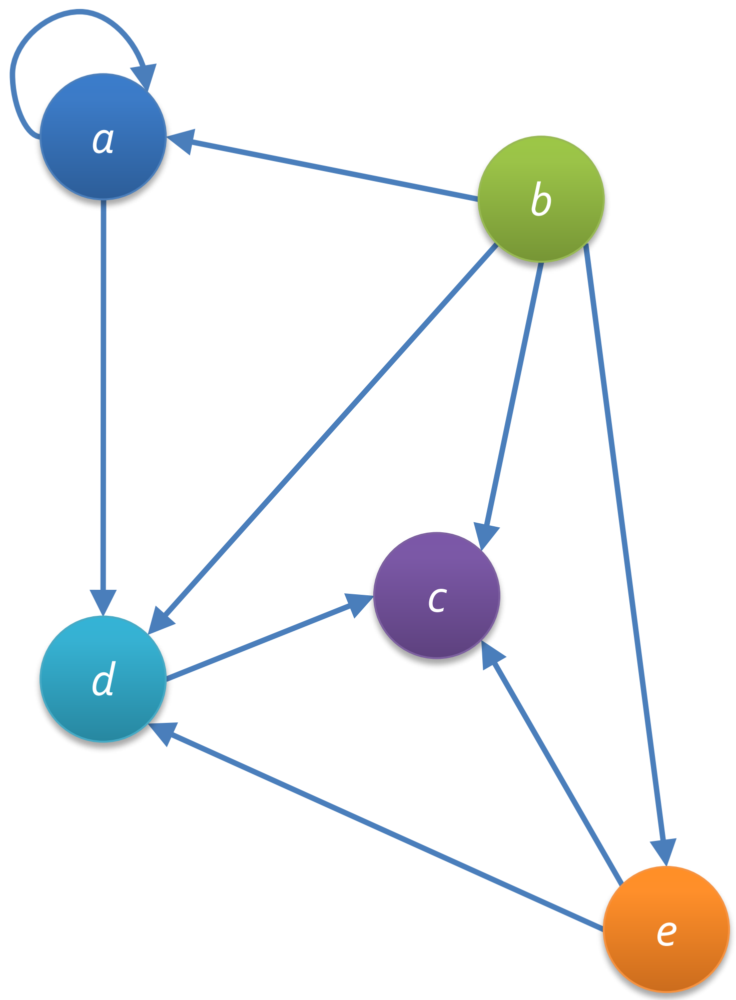
- Consider the sets $A = \{2, 4, 6\}$ and $B = \{1, 3, 5\}$
- Let R be the relation $\{(2,5), (4,1), (4,3), (6,5)\}$
  - Draw the arrow diagram for R
  - Is R a function?
- Let S be the relation for all $(x, y) \in A \times B$, $(x, y) \in S$ iff $y = x + 1$
  - Draw the arrow diagram for S
  - Is S a function?
- $x^2 + y^2 = 1$ on real numbers is not a function for both reasons

Inverses

- We've relaxed things considerably by moving from functions to relations
- All relations have inverses (just reverse the order of the ordered pairs)
- Example
  - Let $A = \{2,3,4\}$ and $B = \{2,6,8\}$
  - For all $(x, y) \in A \times B$, $x\ R\ y \leftrightarrow x \mid y$
  - List the ordered pairs in R
  - List the ordered pairs in R^{-1}

Directed graphs

- A directed graph describes a relationship between nodes
- One way to record a graph is as a matrix
- We can also think of a directed graph as a relation from a set to itself
- What's the relation for this directed graph?



Properties of Relations

Reflexive

- We will discuss many useful properties that hold for any binary relation R on a set A (that is from elements of A to other elements of A)
- Relation R is **reflexive** iff for all $x \in A, (x, x) \in R$
- Informally, *R* is reflexive if every element is related to itself
- *R* is **not** reflexive if there is an $x \in A$, such that $(x, x) \notin R$

Symmetric

- Relation R is **symmetric** iff for all $x, y \in A$, if $(x, y) \in R$ then $(y, x) \in R$
- Informally, R is symmetric if for every element related to another element, the second element is also related to the first
- R is **not** symmetric if there is an $x, y \in A$, such that $(x, y) \in R$ but $(y, x) \notin R$

Transitive

- Relation R is **transitive** iff for all $x, y, z \in A$, if $(x, y) \in R$ and $(y, z) \in R$ then $(x, z) \in R$
- Informally, R is transitive when an element is related to a second element and a second element is related to a third, then it must be the case that the first element is also related to the third
- R is **not** transitive if there is an $x, y, z \in A$, such that $(x, y) \in R$ and $(y, z) \in R$ but $(x, z) \notin R$

Examples

- Let $A = \{0, 1, 2, 3\}$
- Let $R = \{(0,0), (0,1), (0,3), (1,0), (1,1), (2,2), (3,0), (3,3)\}$
  - Is R reflexive?
  - Is R symmetric?
  - Is R transitive?
- Let $S = \{(0,0), (0,2), (0,3), (2,3)\}$
  - Is S reflexive?
  - Is S symmetric?
  - Is S transitive?
- Let $T = \{(0,1), (2,3)\}$
  - Is T reflexive?
  - Is T symmetric?
  - Is T transitive?

Transitive closures

- If you have a set A and a binary relation R on A, the **transitive closure** of R called R^t satisfies the following properties:
  - R^t is transitive
  - $R \subseteq R^t$
  - If S is any other transitive relation that contains R, then $R^t \subseteq S$
- Basically, the transitive closure just means adding in the least amount of stuff to R to make it transitive
- If you can get there in R, you can get there **directly** in R^t

Transitive closure example

- Let $A = \{0, 1, 2, 3\}$
- Let $R = \{(0,1), (1,2), (2,3)\}$
- Is R transitive?
- Then, find the transitive closure of R

Examples on real sets

- Let R be a relation on real numbers $\mathbb{R}$ such that
  - $x\ R\ y \leftrightarrow x = y$
  - Is R reflexive?
  - Is R symmetric?
  - Is R transitive?
- Let S be a relation on real numbers $\mathbb{R}$ such that
  - $x\ S\ y \leftrightarrow x < y$
  - Is S reflexive?
  - Is S symmetric?
  - Is S transitive?
- Let T be a relation on positive integers $\mathbb{N}$ such that
  - $m\ T\ n \leftrightarrow 3 \mid (m - n)$
  - Is T reflexive?
  - Is T symmetric?
  - Is T transitive?

Equivalence Relations

Partitions

- A partition of a set A (as we discussed earlier) is a collection of nonempty, mutually disjoint sets, whose union is A
- A relation can be induced by a partition
- For example, let $A = \{0, 1, 2, 3, 4\}$
- Let A be partitioned into $\{0, 3, 4\}, \{1\}, \{2\}$
- The binary relation induced by the partition is: $x \; R \; y \leftrightarrow x$ and y are in the same subset of the partition
- List the ordered pairs in R

Equivalence relations

- Given set A with a partition
- Let R be the relation induced by the partition
- Then, R is reflexive, symmetric, and transitive
- As it turns out, **any** relation R is that is reflexive, symmetric, and transitive induces a partition
- We call a relation with these three properties an **equivalence relation**

Congruences

- We say that m is congruent to n modulo d if and only if $d \mid (m - n)$
- We write this:
  - $m \equiv n \pmod{d}$
- Congruence mod d defines an equivalence relation
  - Reflexive, because $m \equiv m \pmod{d}$
  - Symmetric because $m \equiv n \pmod{d}$ means that $n \equiv m \pmod{d}$
  - Transitive because $m \equiv n \pmod{d}$ and $n \equiv k \pmod{d}$ mean that $m \equiv k \pmod{d}$
- Which of the following are true?
  - $12 \equiv 7 \pmod{5}$
  - $6 \equiv -8 \pmod{4}$
  - $3 \equiv 3 \pmod{7}$

Equivalence classes

- Let A be a set and R be an equivalence relation on A
- For each element a in A, the **equivalence class of** a, written $[a]$, is the set of all elements x in A such that $a\ R\ x$
- Example
  - Let A be $\{0, 1, 2, 3, 4, 5, 6, 7, 8\}$
  - Let R be congruence $\bmod 3$
  - What's the equivalence class of 1?
- For A with R as an equivalence relation on A
  - If $b \in [a]$, then $[a] = [b]$
  - If $b \notin [a]$, then $[a] \cap [b] = \emptyset$

Ticket Out the Door

Upcoming

Next time…

- Finish equivalence relations
- Modular arithmetic
- Partial orders
- Review for Exam 2

Reminders

- **Work on Assignment 3**
  - **Due Friday**
- **No office hours on Wednesday, Thursday, or Friday**
- **No class on Friday**
- Read 8.4 and 8.5